The Fregean Axiom and Polish Mathematical Logic in the 1920s.

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Man, so far as I know, is the only animal capable of lying to himself.

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The semantical assumption that all true (and, similarly, all false) sentences describe the same, i.e., have a common referent (Bedeutung) is called the Fregean Axiom. Another formulation of it is the relacement law of materially equivalents salva veritate. It also has several ontological versions, e.g.,

\[(\forall p \forall q ((p \ equival q) \Rightarrow (\Phi(p) \ equival \Phi(q))))\]

The main thesis of this talk is twofold:

1. The construction of so called many-valued logics by Jan Łukasiewicz was the effective abolition of the Fregean Axiom;

2. Łukasiewicz is the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic to the present day.

The story I am going to tell you started very early, just before 1910. But on the other hand, my presentation of it is based on certain results originated with Lindenbaum and Tarski and, dated much later, just before 1930.

Lindenbaum and Tarski observed that the formalized language \(\mathcal{L}\) is an absolutely free or anarchic algebraic structure and, hence, the fountain of the whole class \(K(\mathcal{L})\) of all algebraic structures similar to \(\mathcal{L}\). The connections between \(\mathcal{L}\) and any structure \(\mathcal{A}\) in \(K(\mathcal{L})\) are given by maps of \(\mathcal{L}\) to \(\mathcal{A}\) satisfying so called morphism conditions and labelled here as algebraic valuations of \(\mathcal{L}\) over \(\mathcal{A}\). They are admissible reference assignments. The domain of them consists of all expressions of definite syntactic category: formulas (sentences), terms (names) and diverse kinds of formators. The size of codomains of algebraic valuations is not a priori limited. In particular, the formulas may have many algebraic values (admissible referents).

Algebraic structures in \(K(\mathcal{L})\) or, in a suitably chosen subclass of \(K(\mathcal{L})\) when supplied with “distinguished” sets of admissible referents of for-
formulas turn into models and, Tarski was able, making use of algebraic valuations to define the algebraic notions of satisfaction, truth and consequence, relating to formulas of $\mathcal{L}$.

Tarski created also the theory of inference relations (finitary consequence operations) on the set of all formulas. Consequently, in case of any logic considered as an inference relation $\models$, compare [1], one can find sets $V$ of zero-one valued functions defined for all formulas and, called here logical valuations, with the following adequacy property: where $a_1, \ldots, a_n, b$ are any formulas with $n = 0, 1, 2, \ldots$ then:

$$a_1, \ldots, a_n \models b \text{ if and only if for all } t \in V,$$

$$t(b) = 1 \text{ whenever } t(a_1) = \ldots = t(a_n) = 1.$$  

In short, every logic is (logically) two-valued. This general statement can be easily exemplified in case of Łukasiewicz's three-valued sentential logic, $\mathcal{L}_3$.

The adequate sets $V$ form an interval ($V_1 \subseteq V \subseteq V_2$) between the smallest adequate set $V_1$ and the largest one $V_2$. Some adequate sets are better, some other are worse. In general, each adequate set of logical valuations bears a natural (dual) pre-topology which turns out to be a genuine topology in certain important cases.

On the other hand, the logical valuations are morphisms (of formulas to the zero-one model) in some exceptional cases, only.

Thus, the logical valuations and algebraic valuations are functions of quite different conceptual nature. The former relate to the truth and falsity and, the latter represent the reference assignments. The formulas play a double semantical role, in general. It is the Fregean Axiom which amalgamates it into the inseparable unity.

Now, the proof of our thesis runs as follows. In case of the truth-functional logic, the logical valuations and algebraic valuations coincide (in accord with the Fregean Axiom) and are represented by 1 and 0. Clearly, then, the material equivalence is the identity connective. Obviously, any multiplication of logical values is a mad idea and, in fact, Łukasiewicz did not actualize it. Indeed, he defined his three-valued logic $\mathcal{L}_3$ making essential use of algebraic valuations to a suitable algebra on the three element set $\{1, 1/2, 0\}$ with 1 as the sole "distinguished" element. Actually, Łukasiewicz defined but the logical system of $\mathcal{L}_3$-tautologies. However, one may reformulate $\mathcal{L}_3$ in a natural way as an inference relation (compare [2]). Then, the following features of Łukasiewicz's logic $\mathcal{L}_3$ can be revealed [3]:

(a) $\mathcal{L}_3$ is logically two-valued as stated previously.
(b) $\mathcal{L}_3$ is a classical logic in the sense of [4].
(c) $\mathcal{L}_3$ is a particular strengthening of SCI, i.e., the sentential calculus with identity (compare [5], [6]).
The algebraic values 1, 1/2, 0 are, by no means, the logical values typical for \( L_3 \). They represent rather referents of formulas, admissible by \( L_3 \). One may also say that Łukasiewicz constructed the logic \( L_3 \) under the assumption (contradicting to the Fregean Axiom) that there exist exactly three admissible referents of formulas. One of them obtains and the other two do not.

Thus, the Fregean Axiom has been constructively abolished by Jan Łukasiewicz. However, he did not as he could not, create any new logical value besides the truth and falsity. To be sure, POSSIBILITY is our only hope and the headspring of all our failures. It is, however, neither a logical value nor what formulas may refer to.

Because of Łukasiewicz's unusual personality, the possibility and creative freedom were his dearest intellectual idols. But, how could he confuse the truth and falsity with what the sentences describe? How was it possible that the humbug of many logical values persisted over the last fifty years?

To many logicians the problem may seem to be quite simple: Łukasiewicz's affair consists in a shift in technical terminology. This is true but it is not the whole truth, I think. The Polish school in logic was both philosophical and mathematical. It was a pioneer in so called modern scientific philosophy (logical empiricism of some kind) and also had a good sense of problems of classical philosophy. On the other hand, the growing set-theoretical thinking of Polish mathematicians exerted a considerable pressure on minds of their fellows in logic. Many factors contributed to the intense intellectual movement in Polish logic at these times. Consequently, the terminology changed tendentiously in a quite unconventional way. Certainly, it was an abuse of words because the semantical duplicity of formulas disappeared eventually and, after 50 years we still face an illogical paradise of many truths and falsehoods.

The Polish logical school in the twenties was a phenomenon which calls for a deeper analysis along the lines of the history of ideas. At present, four spicy items of virtually formal nature should be noted (compare [7], [8]).

First of all, in the early twenties Łukasiewicz used to equate material equivalence with identity. He did openly so even with respect to the equivalence connective of his logic \( L_3 \).

Secondly, in his doctoral dissertation (1923) Alfred Tarski states the independence of the Fregean Axiom \( AF \). In fact, Tarski explicitly compared \( AF \) with the fifth postulate of Euclidean Geometry and proved the independence of \( AF \) by means of the logic \( L_3 \), that is, \( SCI (!) \).

Thirdly, it was Leśniewski's idea, I conjecture, to eliminate all abbreviations in favour of definitional axioms built by means of the material equivalence. It underlies Tarski's paper. His fundamental theorem (TH. 11) is a material equivalence considered by him and, generally recognized
to day as a definition of conjunction. Similarly, several material equi-
valencies underlying Tarski's system are explicitly labelled as definitions
(DEF). One may ponder on all those DEFINITIONS in view of the
absence of AF in Tarski's system.

Finally, Stanisław Leśniewski confessedly accepted the Fregean Axiom
and, consequently, his protothetics became transformed into a trivial
theory of two entities 1 and 0, compare [9]. To support his views, Leśniew-
ski claimed to have constructed a general method which makes it possible
to eliminate all functions \( \Phi \) not satisfying AF. One is eager to compare
Leśniewski's method with the independence result by his own pupil.

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